

JOINT AND CONDITIONAL R-NORM INFORMATION MEASURE

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The present paper depicts the joint and conditional probability distribution of two random variables ξ and η having probability distributors P and Q over the Sets $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ respectively. Then the R-norm information of the random variables is denoted by $H_R(\xi) = H_R(P)$ and $H_R(\eta) = H_R(Q)$, where

$$p_i = P(\xi = x_i), i = 1, 2, \dots, n, \quad p_j = P(\eta = y_j), j = 1, 2, \dots, m$$

are the probabilities of the possible values of the random variables. Similarly, we consider a two-dimensional discrete random variable (ξ, η) with joint probability distribution $\pi = (\pi_{11}, \pi_{12}, \dots, \pi_{1n})$,

where $\pi_{ij} = P(\xi = x_i, \eta = y_j)$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ is the joint probability for the values (x_i, y_j) of (ξ, η) . We shall denote conditional probabilities by p_{ij} and q_{ij} such

that $\pi_{ij} = p_{ij} q_j = q_{ji} p_i$ And $p_i = \sum_{j=1}^m \pi_{ij}$ and $q_j = \sum_{i=1}^n \pi_{ij}$.

DEFINITION: The joint R-norm information measure for $R \in R^+$ and is given by

$$H_R(\xi, \eta) = \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] \quad (1.1)$$

Proposition 1: $H_R(\xi, \eta)$ is symmetric in ξ and η .

Proof: The joint R-norm information measure is defined by

$$\begin{aligned} H_R(\xi, \eta) &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m P^R(\xi = x_i, \eta = y_j) \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m P^R(\xi = x_i) P^R(\eta = y_j) \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m P^R(\eta = y_j) P^R(\xi = x_i) \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m P^R(\eta = y_j, \xi = x_j) \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ji}^R \right\}^{\frac{1}{R}} \right] = H_R(\eta, \xi) \end{aligned}$$

This implies that $H_R(\xi, \eta)$ is symmetric in ξ, η .

Proposition 2: If ξ and η are stochastically independent. Then the following holds

$$H_R(\xi, \eta) = H_R(\xi) + H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta) \quad (1.2)$$

Proof: Since the joint R-norm information measure for $R \in R^+$ and is given by

$$\begin{aligned} H_R(\xi, \eta) &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m P^R(\xi = x_i, \eta = y_j) \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m P^R(\xi = x_i) P^R(\eta = y_j) \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m P^R(\xi = x_i) P^R(\eta = y_j) \right\}^{\frac{1}{R}} \right] \end{aligned} \quad (1.3)$$

Since ξ and η are stochastically independent, thus (1.3) becomes

$$\begin{aligned} H_R(\xi, \eta) &= \frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n P^R(\xi = x_i) \right]^{\frac{1}{R}} \cdot \left[\sum_{j=1}^m P^R(\eta = y_j) \right]^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} - \frac{R}{R-1} \left[\left(1 - \frac{R-1}{R} H_R(\xi) \right) \left(1 - \frac{R-1}{R} H_R(\eta) \right) \right] \\ &= H_R(\xi) + H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta) \quad \text{Thus finally} \end{aligned}$$

$$H_R(\xi, \eta) = H_R(\xi) + H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta) \quad (1.4)$$

In the limiting case $R \rightarrow 1$ we find the additive form of Shannon's information measure for independent random variables. i.e. when $R \rightarrow 1$ in (1.4), then we get

$$H_R(\xi, \eta) = H_R(\xi) + H_R(\eta) - \frac{1-1}{1} H_R(\xi) H_R(\eta), \Rightarrow H_R(\xi, \eta) = H_R(\xi) + H_R(\eta)$$

To construct a conditional R-norm information measure we can use a direct and an indirect method. The indirect method leads to next definition

DEFINITION: The average subtractive conditional R-norm information of η given ξ for $R \in R^+$ and is defined as

$$\begin{aligned} {}^\delta H_R(\eta / \xi) &= H_R(\xi, \eta) - H_R(\xi) \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] - \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} \right] \end{aligned} \quad (1.4)$$

$$= \frac{R}{R-1} \left[\left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] \quad (1.5)$$

$$= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] - \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} \right] \quad (1.6)$$

$$= H_R(\eta / \xi) + H_R(\xi)$$

Thus $H_R(\xi, \eta) = H_R(\eta / \xi) + H_R(\xi)$

A direct way to construct a conditional R-norm information is the following.

DEFINITION: The average conditional R-norm information of η given

ξ is for $R \in R^+$ defined as

$$^* H_R(\eta / \xi) = \frac{R}{R-1} \left[1 - \sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \quad (1.7)$$

Or alternatively

$$^{**} H_R(\eta / \xi) = \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \quad (1.8)$$

The two conditional measure given in (1.7) and (1.8) differ by the way the probabilities p_i are incorporated. The expression (1.7) is a true mathematical expression over ξ , whereas the expression (1.8) is not.

Theorem: If ξ and η are statistically independent random variables then for $R \in R^+$

$$(1) \quad {}^{\delta}H_R(\eta/\xi) = \frac{R}{R-1} \left[\left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} - \left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} \cdot \left\{ \sum_{j=1}^m q_j^R \right\}^{\frac{1}{R}} \right]$$

$$(2) \quad {}^{\delta}H_R(\eta/\xi) = H_R(\xi, \eta) - H_R(\xi) = H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta)$$

$$(3) \quad {}^*H_R(\eta/\xi) = H_R(\eta)$$

$$(4) \quad {}^{**}H_R(\eta/\xi) = H_R(\eta)$$

Proof: (I) Since the average subtractive conditional R-norm information of η given ξ is for $R \in R^+$ defined as

$${}^{\delta}H_R(\eta/\xi) = \frac{R}{R-1} \left[\left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] \quad (1.9)$$

Substitute $\pi_{ij} = p_{ij} q_j$ in (1.9), we get

$${}^{\delta}H_R(\eta/\xi) = \frac{R}{R-1} \left[\left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} - \left\{ \sum_{i=1}^n \sum_{j=1}^m (p_{ij} q_j)^R \right\}^{\frac{1}{R}} \right] \quad (1.10)$$

Since ξ and η are stochastically independent. Thus (1.10) becomes

$${}^{\delta}H_R(\eta/\xi) = \frac{R}{R-1} \left[\left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} - \left\{ \sum_{i=1}^n p_i^R \right\}^{\frac{1}{R}} \cdot \left\{ \sum_{j=1}^m q_j^R \right\}^{\frac{1}{R}} \right]$$

(II) Since we know that if ξ and η are stochastically independent. Then the following holds

$$\begin{aligned} H_R(\xi, \eta) &= H_R(\xi) + H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta) \\ \Rightarrow H_R(\xi, \eta) - H_R(\xi) &= H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta) \end{aligned} \quad (1.11)$$

And we know

$${}^\delta H_R(\eta / \xi) = H_R(\xi, \eta) - H_R(\xi) \quad (1.12)$$

Using (1.12) in (1.11), we get

$${}^\delta H_R(\eta / \xi) = H_R(\xi, \eta) - H_R(\xi) = H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta)$$

(III) Since the average conditional R-norm information of η given ξ for $R \in R^+$ and is defined as

$${}^* H_R(\eta / \xi) = \frac{R}{R-1} \left[1 - \sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji} \right\}^{\frac{1}{R}} \right] \quad (1.13)$$

Substitute $q_{ji} = q_j$ in (1.13), we get

$$\begin{aligned} {}^*H_R(\eta/\xi) &= \frac{R}{R-1} \left[1 - \sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_j^R \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{j=1}^m q_j^R \right\}^{\frac{1}{R}} \right] = H_R(\eta) \quad \left[\because \left[\sum_{i=1}^n p_i \right] = 1 \right] \end{aligned}$$

$$\text{Hence } {}^*H_R(\eta/\xi) = H_R(\eta) \quad (1.14)$$

(IV) Since the average conditional R-norm information of η given ξ for $R \in R^+$ and is defined as

$${}^{**}H_R(\eta/\xi) = \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \quad (1.15)$$

Substitute $q_{ji} = q_j$ in (1.15), we get

$$\begin{aligned} {}^{**}H_R(\eta/\xi) &= \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^n p_i \sum_{j=1}^m q_j^R \right\}^{\frac{1}{R}} \right] \\ &= \frac{R}{R-1} \left[1 - \left\{ \sum_{j=1}^m q_j^R \right\}^{\frac{1}{R}} \right] = H_R(\eta) \quad \left[\because \left[\sum_{i=1}^n p_i \right] = 1 \right] \end{aligned}$$

Hence ${}^{**}H_R(\eta/\xi) = H_R(\eta)$

From this theorem we may conclude that the measure ${}^{\delta}H_R(\eta/\xi)$, which is obtained by the formal difference between the joint and the marginal information measure, does not satisfy requirement (I). Therefore it is less attractive than the two other measure. In the next theorem we consider requirement (II), for the conditional information measures ${}^*H_R(\eta/\xi)$ and ${}^{**}H_R(\eta/\xi)$.

Theorem: If ξ and η are discrete random variables then for $R \in R^+$ then the following results hold.

$$(I) \quad {}^*H_R(\eta/\xi) \leq H_R(\eta) \quad (II) \quad {}^{**}H_R(\eta/\xi) \leq H_R(\eta)$$

$$(III) \quad {}^{**}H_R(\eta/\xi) \leq {}^*H_R(\eta/\xi) \quad (IV) \quad {}^{**}H_R(\eta/\xi) \leq {}^*H_R(\eta/\xi) \leq H_R(\eta)$$

The equality signs holds if ξ and η are independent. **Proof:**

(I) Here we consider two cases: Cases I: when $R < 1$

We know by [4] that for $R > 1$.

$$\left[\sum_{j=1}^m \left\{ \sum_{i=1}^n X_{ij} \right\}^R \right]^{\frac{1}{R}} \leq \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m X_{ij}^R \right\}^{\frac{1}{R}} \right] \quad (1.16)$$

Setting $x_{ij} = \pi_{ij} \geq 0$ in (1.16), we have

$$\left[\sum_{j=1}^m \left\{ \sum_{i=1}^n \pi_{ij} \right\}^R \right]^{\frac{1}{R}} \leq \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] \quad (1.17)$$

$$\text{Since } q_i = \sum_{j=1}^m \pi_{ij} \text{ and } \pi_{ij} = p_i q_{ji} \quad (1.18)$$

Using (1.18) in (1.17), we get

$$\left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \leq \left[\sum_{i=1}^n \sum_{j=1}^m (q_{ji} p_i)^R \right]^{\frac{1}{R}} \quad (1.19)$$

It can be written as

$$\begin{aligned} \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} &\leq \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \\ - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} &\geq - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \\ 1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} &\geq 1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \end{aligned} \quad (1.20)$$

We know $\frac{R}{R-1} > 0$ if $R > 1$

Multiplying both sides of (1.20) by $\frac{R}{R-1}$, we get

$$\frac{R}{R-1} \left[1 - \left[\sum_{j=1}^n q_j^R \right]^{\frac{1}{R}} \right] \geq \frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \right] \quad (1.21)$$

But $\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \right] = {}^+ H_R(\eta / \xi)$ and

$$\frac{R}{R-1} \left[1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \right] = H_R(\eta) \text{ Thus (1.21) becomes}$$

$${}^* H_R(\eta / \xi) \leq H_R(\eta) \quad \text{for } R > 1 \quad (1.22)$$

Cases II: when $0 < R < 1$

We know by [4] that for $0 < R < 1$

$$\left[\sum_{j=1}^m \left\{ \sum_{i=1}^n x_{ij} \right\}^R \right]^{\frac{1}{R}} \geq \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m x_{ij}^R \right\}^{\frac{1}{R}} \right] \quad (1.23)$$

Setting $x_{ij} = \pi_{ij} \geq 0$ in (1.23), we have

$$\left[\sum_{j=1}^m \left\{ \sum_{i=1}^n \pi_{ij} \right\}^R \right]^{\frac{1}{R}} \geq \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m \pi_{ij}^R \right\}^{\frac{1}{R}} \right] \quad (1.24)$$

Since $q_i = \sum_{j=1}^m \pi_{ij}$

Thus (1.24) becomes

$$\begin{aligned} \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} &\geq \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m (q_{ji} p_i)^R \right\}^{\frac{1}{R}} \right] \\ - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} &\leq - \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m (q_{ji} p_i)^R \right\}^{\frac{1}{R}} \right] \\ \Rightarrow 1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} &\leq 1 - \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m (q_{ji} p_i)^R \right\}^{\frac{1}{R}} \right] \end{aligned} \quad (1.25)$$

We know $\frac{R}{R-1} < 0$ if $0 < R < 1$

Multiplying both sides of (1.25) by $\frac{R}{R-1}$, we get

$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n q_j^R \right]^{\frac{1}{R}} \right] \geq \frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \right] \quad (1.26)$$

But
$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \right] = H_R(\eta / \xi) \quad \text{and}$$

$$\frac{R}{R-1} \left[1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \right] = H_R(\eta)$$

Thus (1.26) becomes

$$\Rightarrow^* H_R(\eta / \xi) \leq H_R(\eta) \quad \text{for } 0 < R < 1. \quad (1.27)$$

Thus from (1.22) and (1.27), we get

$$^* H_R(\eta / \xi) \leq H_R(\eta) \quad \text{for } R \in \mathbb{R}^+$$

(II) Here we consider two cases

Cases I: when $R > 1$

From Jensen's inequality for $R > 1$, we find

$$\sum_{i=1}^n p_i q_{ji}^R \geq \left[\sum_{i=1}^n p_i q_{ji} \right]^R = q_j^R \quad (1.28)$$

After summation over j and raising both sides of (1.28) by power $\frac{1}{R}$, we have

$$\begin{aligned} \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} &\geq \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \\ - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} &\leq - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \\ 1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} &\leq 1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \end{aligned} \quad (1.29)$$

Using $\frac{R}{R-1} > 0$ as $R > 1$, Thus (1.29) becomes

$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \right] \leq \frac{R}{R-1} \left[1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \right] \quad (1.30)$$

$$\text{But } \frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \right] = {}^{++} H_R(\eta / \xi)$$

$$\text{And } \frac{R}{R-1} \left[1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \right] = H_R(\eta)$$

Thus (1.30) becomes

$$^{**}H_R(\eta/\xi) \leq H_R(\eta) \quad \text{for } R > 1 \quad (1.31)$$

Case II: when $0 < R < 1$

From Jensen's inequality for $0 < R < 1$ we find

$$\sum_{i=1}^n p_i q_{ji}^R \leq \left[\sum_{i=1}^n p_i q_{ji} \right]^R = q_j^R \quad (1.32)$$

After summation over j and raising both sides of (1.32) by power $\frac{1}{R}$, we have

$$\begin{aligned} \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} &\leq \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \\ - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} &\geq - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \\ 1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} &\geq 1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \end{aligned} \quad (1.33)$$

Using $\frac{R}{R-1} < 0$ as $0 < R < 1$, Thus (1.33) becomes

$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \right] \leq \frac{R}{R-1} \left[1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \right] \quad (1.34)$$

But
$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \right] = {}^{++} H_R(\eta / \xi)$$

And
$$\frac{R}{R-1} \left[1 - \left[\sum_{j=1}^m q_j^R \right]^{\frac{1}{R}} \right] = H_R(\eta)$$

Thus (1.34) becomes

$${}^{**} H_R(\eta / \xi) \leq H_R(\eta) \quad \text{for} \quad 0 < R < 1 \quad (1.35)$$

Thus from (1.31) and (1.35), we get

$${}^{**} H_R(\eta / \xi) \leq H_R(\eta) \quad \text{for} \quad R \in \mathbb{R}^+$$

(III) Here we consider two cases:

Cases I: when $R > 1$

We know from Jensen's inequality

$$\left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \leq \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \quad \text{for } R > 1 \quad (1.36)$$

$$-\left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \geq -\left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}}$$

$$1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \geq 1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \quad (1.37)$$

Using $\frac{R}{R-1} > 0$ if $R > 1$, then (1.37) becomes

$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \right] \geq \frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \right] \quad (1.38)$$

But
$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \right] = {}^+ H_R(\eta / \xi)$$

And
$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \right] = {}^{++} H_R(\eta / \xi)$$

Thus (1.38) becomes ${}^{**} H_R(\eta / \xi) \leq {}^* H_R(\eta / \xi)$ for $R > 1$

(1.39) **Case II: when $0 < R < 1$** We know from Jensen's inequality

$$\left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \geq \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \quad \text{for } 0 < R < 1 \quad (1.40)$$

$$-\left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \leq -\left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}}$$

$$1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \leq 1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \quad (1.41)$$

Using $\frac{R}{R-1} < 0$ if $0 < R < 1$, then (3.41) becomes

$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \right] \geq \frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \right] \quad (1.42)$$

But
$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^R \right\}^{\frac{1}{R}} \right] \right] = {}^+ H_R(\eta / \xi)$$

And
$$\frac{R}{R-1} \left[1 - \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^R \right]^{\frac{1}{R}} \right] = {}^{++} H_R(\eta / \xi)$$
 Thus (1.42) becomes

$${}^{**} H_R(\eta / \xi) \leq {}^* H_R(\eta / \xi) \quad \text{for } 0 < R < 1 \quad (1.43)$$

Thus from (1.39) and (1.43), we get ${}^{**} H_R(\eta / \xi) \leq {}^* H_R(\eta / \xi)$ for $R \in \mathbb{R}^+$

(IV) From (I) and (III), we have ${}^* H_R(\eta / \xi) \leq H_R(\eta)$ And

${}^{**} H_R(\eta / \xi) \leq {}^* H_R(\eta / \xi)$ Thus finally we find

$${}^{**} H_R(\eta / \xi) \leq {}^* H_R(\eta / \xi) \leq H_R(\eta) \quad (1.44)$$

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